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Multiplicities of angular momenta in a system of N-dimensional oscillators and the reduction $SU(N) \supset O(3)$

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Abstract. The method of enumeration of all O(3) irreducible representations contained in the representation of a system of an arbitrary number of N-dimensional harmonic oscillators with definite quantum number is presented. The close connection of this problem with the reduction $SU(N) \supset O(3)$ is discussed. A new simple graphical method for the determination of the angular momentum content of SU(3) irreducible representations is introduced.

1. Introduction

One of the important problems in group theory and its physical applications is a problem concerning the decomposition of irreducible representation (IR) of $SU(N)$ into IR of O(3). It arises particularly from the angular momentum (AM) classification of states of n-particle systems when each particle has $2s+1 = N$ energy levels.

A most careful study has been made of the reduction $SU(3) \supset O(3)$. For this case various explicit orthogonal and non-orthogonal bases were constructed, relations between different bases and comparisons with the case of the canonical reduction $SU(3) \supset SU(2) \supset U(1)$ were given, etc. An appreciable difficulty in the solution of the reduction $SU(3) \supset O(3)$ was the missing label problem. All these questions were discussed by Elliot (1958a, b), Bargmann and Moshinsky (1960, 1961), Judd *et al* (1974), Moshinsky *et al* (1975), Hughes (1973a, b), Green and Bracken (1974), Green *et al* (1976), Sharp (1975).

On the other hand $SU(N)$ is a symmetry group of the N-dimensional harmonic oscillator (HO) (Baker 1956). This fact has been the starting point for a variety of papers concerning the group theoretical properties of HO (Wybourne 1974 and references therein) and the boson operator realisation of the $SU(N)$ state vectors. But the latter deals especially with the state vectors of the canonical reduction $SU(N) \supset SU(N-1) \supset \ldots U(1)$ (Baird and Biedenharn 1963, Louck 1965, Ciftan and Biedenharn 1969).

In the present paper by means of a boson-operator method we will consider the reduction $SU(N) \supset O(3)$. On the whole we will be interested in the outward side of this reduction: the determination of the multiplicities of the O(3) IR in the $SU(N)$ IR. It is to be noted that earlier Jahn (1950) (see also Flowers 1952), suggested the recurrence method, now widely used, based on decomposition of the outer multiplications of the symmetric group IR. The essence of the method is given for example in Hamermesh (1962) and Kaplan (1969).

Besides that another method for the particular case of this problem was described by Büttner (1967). He has constructed generating functions, recurrence relations and tables for AM multiplicities in the system of bosons each having spin s, which is equivalent to finding the multiplicities of O(3) IR in the SU(N) symmetric IR (the IR which correspond to a Young tableau of one row).

Later, the present author (Mikhailov 1974, to be referred to as I) has obtained analogous formulae by another method, based upon the boson representation of AM in accordance with the methods of Schwinger (1965) and Bargmann (1962).

Below we give the method of enumeration of all O(3) IR contained in the representations of O(3)×O(3) for the system of an arbitrary number of N-dimensional HO (§§ 1,2). Similar to I we introduce auxiliary numbers for the purpose of finding the AM multiplicities in the system and give the recurrence relations and generating function for these numbers (§ 3). The four tables of the multiplicities and their general properties are given (§ 3). The close connection of this problem with the reduction SU(N)⊃O(3) is discussed (§§ 4,5). A new simple graphical method for the determination of the AM content of SU(3) IR is introduced (appendix).

2. Multiplicities of O(3)×O(3) IR contained in the representation of a system of N-dimensional HO

Let us consider the $N.N'$ pairs of the creation and annihilation boson operators a_μ^ν and $\bar{a}_\mu^\nu$ which satisfy the usual commutation relations

$$[\bar{a}_\mu^\nu, a_{\mu'}^{\nu'}] = \delta_{\nu\nu'}\delta_{\mu\mu'}, \qquad [a_\mu^\nu, a_{\mu'}^{\nu'}] = [\bar{a}_\mu^\nu, \bar{a}_{\mu'}^{\nu'}] = 0 \tag{1}$$

$$\begin{aligned} &\mu = -s, -s+1, \ldots s; \qquad && \nu = -s', -s'+1, \ldots s', \\ &N = 2s+1, N' = 2s'+1; && s, s' = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots. \end{aligned} \tag{2}$$

We can say that the operators a_μ^ν describe N' HO each having dimension N. But below, for the sake of brevity, we refer to this system as $N.N'$ HO.

The following quadratic combination:

$$n = \sum_\nu n^\nu = \sum_\mu n_\mu = \sum_{\nu,\mu} a_\mu^\nu \bar{a}_\mu^\nu \tag{3}$$

is the particle number operator and

$$\begin{aligned} &J_+ = \sum_\nu \sum_\mu [(s-\mu)(s+\mu+1)]^{1/2} a_{\mu+1}^\nu \bar{a}_\mu^\nu, \\ &J_0 = \sum_\nu \sum_\mu \mu n_\mu^\nu, \qquad J_- = (J_+)^+, \end{aligned} \tag{4}$$

$$\begin{aligned} &J^+ = \sum_\mu \sum_\nu [(s'-\nu)(s'+\nu+1)]^{1/2} a_\mu^{\nu+1} \bar{a}_\mu^\nu, \\ &J^0 = \sum_\mu \sum_\nu \nu n_\mu^\nu, \qquad J^- = (J^+)^+ \end{aligned} \tag{5}$$

are AM operators of two kinds: with lower and upper indexes. The J operators satisfy AM commutation relations. For example for lower-index operators we have

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0.$$

Any lower-index operator and any upper-index operator have vanishing commutator and both commute with the particle number operator. Therefore from various quadratic combinations we have picked two sets of operators which constitute two independent O(3) algebras. After that we can label the states of $N.N'$ HO by the five quantum numbers n, j, m, τ, ζ, where $j(j+1)$, m are the eigenvalues of $J_0(J_0-1)+J_+J_-$, J_0 and $\tau(\tau+1)$, ζ are the eigenvalues of the analogous operators with upper indices. It is evident that

$$\begin{aligned} &j = ns, ns-1, \ldots, 0 \text{ or } \tfrac{1}{2}, \\ &\tau = ns', ns'-1, \ldots, 0 \text{ or } \tfrac{1}{2}, \\ &m = -j, -j+1, \ldots, j, \qquad \zeta = -\tau, -\tau+1, \ldots, \tau. \end{aligned} \tag{6}$$

Of course this number of indices is not sufficient for the full description of all states even in the simple case of 3.2 HO. Because of this shortage of labels AM multiplicities greater than one appear, and these multiplicities will be considered.

Homogeneous polynomials of degree n in a_μ^ν represent some state of our HO system. Each state or a manifold of the states may be expressed in terms of basis states of IR D_j of the first or the second O(3) group. Our aim is the analysis of possible j multiplicities without special operator construction of the states.

For this purpose we will build graphs, generalising the trees from I to three-dimensional trees. As an example, figure 1 shows a tree for the 3.2 HO. Below we describe the properties of this graph and its method of construction.

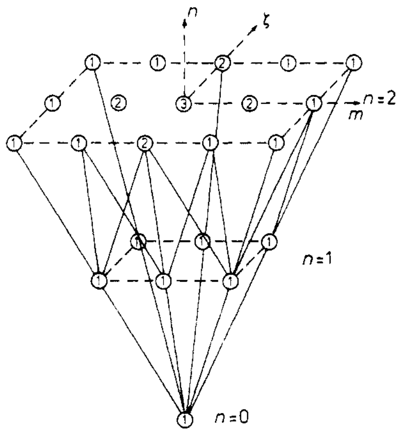


Figure 1. The tree for 3.2 HO. For the sake of clarity some of the ribs are omitted. From each of the vertices corresponding to $n=1$ and $n=2$ run out $3.2=6$ ribs as from the vertex corresponding $n=0$.

Open circles in figure 1 are the vertices. The tree rests upon the ground state vertex and stretches upwards to infinity. Each rib (line connecting two adjacent vertices) corresponds to a definite creation operator a_μ^ν for upwards motion and an annihilation operator $\bar{a}_\mu^\nu$ for downwards motion. From each vertex $N.N'$ ribs run upwards.

Each vertex corresponds to some state of $N.N'$ HO

$$a_{\mu_1}^{\nu_1} a_{\mu_2}^{\nu_2} \ldots a_{\mu_n}^{\nu_n} |0\rangle. \tag{7}$$

All the possible vertices, lying in a horizontal plane, describe the variety of HO states with definite n.

Further the route is a sequence of the ribs when the end of each rib coincides with the beginning of the next rib. We consider the routes escaping from the ground state vertex upwards. The state (7) corresponds to a definite route. Two routes are equivalent if it is possible to get one route from another by means of boson-operator permutations. In each vertex (open circle) we put the number of inequivalent routes $C^{ss'}_{nm\zeta}$. In I the corresponding two-dimensional numbers were P^{m}_{ls} $(l=n)$. The upper indices in $C^{ss'}_{nm\zeta}$ define the type of the tree and the lower ones define the site of the vertex in a given tree.

Let us consider an $\mathrm{O(3)\times O(3)}$ representation $Q^{ss'}_n$. Its representation space is based upon all n-quantum states of $N.N'$ HO. $Q^{ss'}_n$ can be expanded into the $\mathrm{O(3)\times O(3)}$ representations $D_{j\tau}$:

$$Q^{ss'}_n=\sum_{j,\tau} q^{ss'}_{nj\tau} D_{j\tau} \tag{8}$$

where $D_{j\tau}=D_j\otimes D_\tau$; D_j, D_τ are O(3) IR. The dimension $M^{ss'}_n$ of the $Q^{ss'}_n$ representation space is

$$M^{ss'}_n=\binom{n+N.N'-1}{n}. \tag{9}$$

These dimensions satisfy the requirements

$$M^{ss'}_n=\sum_{m=-j}^{j}\sum_{\zeta=-\tau}^{\tau} C^{ss'}_{nm\zeta}, \tag{10}$$

$$M^{ss'}_n=\sum_{j,\tau}(2j+1)(2\tau+1)q^{ss'}_{nj\tau}. \tag{11}$$

In order to determine basis states of representations $D_{j\tau}$ we seek the state $|\psi\rangle$ which satisfies simultaneously two conditions: $J_+|\psi\rangle=0$ and $J^+|\psi\rangle=0$; therefore $|\psi\rangle=|n,j,j,\tau,\tau\rangle$. After that we can construct all other states

$$|n,j,m,\tau,\zeta\rangle\propto(J_+)^{j-m}(J^+)^{\tau-\zeta}|n,j,j,\tau,\tau\rangle$$

of this basis.

The multiplicities $q^{ss'}_{nj\tau}$ and numbers $C^{ss'}_{nm\tau}$ are non-negative integers. It was shown in I that for the case $s'=0$ there is the relation

$$q^{s}_{nj}=C^{s}_{nm}-C^{s}_{n,m+1}\big|_{m=j} \tag{12}$$

which agrees with (10) and (11). In order to find a relation generalising (12) to the case of arbitrary s' we consider a concrete example. Suppose $N=3$, $N'=2$, then we can get the verification from table 2 (Bargmann and Moshinsky 1960). The tree from figure 1 corresponds to this case. We put $n=3$. Because of symmetry $C_{m\zeta}=C_{-m,\zeta}=C_{m,-\zeta}$ it is sufficient to write down a quarter of the plane $n=3$:

$$\begin{array}{c|cccc} \zeta & & & & \\ \frac{3}{2} & 2 & 2 & 1 & 1 \\ \frac{1}{2} & 4 & 4 & 2 & 1 \\ \hline & 0 & 1 & 2 & 3 \quad m \end{array} \quad . \tag{13}$$

To extract the basis states, we begin from the state $|j\tau\rangle = |3, \frac{3}{2}\rangle = (a_1^{1/2})^3|0\rangle$ which is unique. From this state we construct all $(2j+1)(2\tau+1) = 7.4 = 28$ basis states $|3, m, \frac{3}{2}, \zeta\rangle$. Subtraction of this basis from the basis of representation $Q_3^{1,1/2}$ corresponds to the subtraction of one from each number in (13). After this subtraction (13) takes the form

$$\begin{array}{c|ccc} \zeta & & & \\ \frac{3}{2} & 1 & 1 & \\ \frac{1}{2} & 3 & 3 & 1 \\ \hline & 0 & 1 & 2 \quad m \end{array} \quad . \tag{14}$$

Now we construct the state $|j, \tau\rangle = |1, \frac{3}{2}\rangle$ and the basis $|1, m, \frac{3}{2}, \zeta\rangle$. Subtracting this basis from the space described in (14) we must subtract one from each number which lies not to the right of and not above $m = 1$, $\zeta = \frac{3}{2}$. The remainder has the form

$$\begin{array}{c|ccc} \zeta & & & \\ \frac{1}{2} & 2 & 2 & 1 \\ \hline & 0 & 1 & 2 \quad m \end{array} \quad . \tag{15}$$

Similarly we extract from (15) two bases $|2, m, \frac{1}{2}, \zeta\rangle$ and $|1, m, \frac{1}{2}, \zeta\rangle$. Now the full decomposition of Q is

$$Q_3 = D_{3,3/2} + D_{1,3/2} + D_{2,1/2} + D_{1,1/2}. \tag{16}$$

The general rule for finding $q_{nj\tau}$ is as follows. From the line τ in the table of numbers $C_{mn\zeta}$ (here we put $m = j$, $\zeta = \tau$) it is necessary to subtract the line $\tau + 1$, obtaining in this way the sequence of $2sn + 1$ numbers. Then we must subtract the number $j + 1$ from the number j in the sequence. The following formula summarises these rules:

$$q_{nj\zeta} = C_{nm\zeta} + C_{n,m+1,\zeta+1} - C_{n,m+1,\zeta} - C_{nm,\zeta+1}\Big|_{\substack{m=j \\ \zeta=\tau}}. \tag{17}$$

So with the help of (17) the problem (8), concerning decomposition of Q_n, is reduced to the determination of route numbers $C_{nm\zeta}$ in the $N.N'$ tree. It should be noted that the four outward planes of the three-dimensional tree coincide with the flat trees from I for which the generating function and the recurrence relations were constructed and for which a number of typical properties have been described.

3. Route numbers $C_{nm\zeta}^{ss'}$

As in I in order to describe the route numbers we will use an analogy with a simple combinatorial problem about the change of money (Polya 1956).

Let us bring a combinatorial equivalent $x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu}$ in correspondence with the operator a_μ^ν and form the product of the infinite series

$$\prod_{\mu,\nu} [1 + x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu} + (x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu})^2 + \ldots]$$

$$= \prod_{\mu,\nu} (1 - x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu})^{-1}. \tag{18}$$

Picking out of each square bracket in (18) only one term and multiplying these we find

$$\prod_{\mu,\nu} (x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu})^{n^\nu_\mu}$$
$$= x^{\Sigma n^\nu_\mu(s+\mu)}y^{\Sigma n^\nu_\mu(s-\mu)}u^{\Sigma n^\nu_\mu(s'+\nu)}v^{\Sigma n^\nu_\mu(s'-\nu)}$$
$$= x^{ns+m}y^{ns-m}u^{ns'+\zeta}v^{ns'-\zeta}. \tag{19}$$

In (18) and (19) μ and ν in the sums and in the products vary in accordance with (2). Expression (19) enters into the product (18) exactly $C_{nm\zeta}$ times. This then implies that

$$\prod_{\mu,\nu} (1-x^{s+\mu}y^{s-\mu}u^{s'+\nu}v^{s'-\nu})^{-1}$$
$$= \sum_{n=0}^{\infty}\sum_{m=-ns}^{ns}\sum_{\zeta=-ns}^{ns} C^{ss'}_{nm\zeta}x^{ns+m}y^{ns-m}u^{ns'+\zeta}v^{ns'-\zeta}. \tag{20}$$

It is this generating function which in general completely defines the route numbers. However, it is easier to determine the numbers with the help of recurrence relations which are consequences of (20).

To get the simple relations we put $s'=\frac{1}{2}$. For the sake of brevity, the index s' in C is omitted. Let us transform (20) in the following way:

$$\sum_{n,m,\zeta} C^s_{nm\zeta}x^{ns+m}y^{ns-m}u^{\frac{1}{2}n+\zeta}v^{\frac{1}{2}n-\zeta}$$
$$= \prod_{\mu,\nu}(1-x^{s+\mu}y^{s-\mu}u^{\frac{1}{2}+\nu}v^{\frac{1}{2}-\nu})^{-1}$$
$$= (1-x^{2s}u)^{-1}(1-x^{2s}v)^{-1}\prod_{\mu,\nu}(1-x^{s_1+\mu}y^{s_1-\mu}u^{\frac{1}{2}+\nu}v^{\frac{1}{2}-\nu})^{-1}$$
$$= (1-x^{2s}u)^{-1}(1-x^{2s}v)^{-1}\sum_{n,m,\zeta} C^{s_1}_{nm\zeta}x^{ns_1+m}y^{ns_1-m}u^{\frac{1}{2}n+\zeta}v^{\frac{1}{2}n-\zeta}, \tag{21}$$

where $s_1 = s-\frac{1}{2}$, $x_1 = xy^{1/(2s-1)}$, $y_1 = y^{2s/(2s-1)}$. Choosing from this chain of equalities the first and the last lines, we obtain the relation between C^s and $C^{s-\frac{1}{2}}$. Comparing the coefficients of $x^ay^bu^cv^d$ on both sides, we find a recurrence formula:

$$C^s_{nm\zeta} = C^{s-\frac{1}{2}}_{n,m+\frac{1}{2}n,\zeta} + \sum_{\nu=-1/2}^{1/2} C^s_{n-1,m-s,\zeta+\nu} - C^s_{n-2,m-2s,\zeta}. \tag{22}$$

Similarly we obtain a recurrence formula for $s'=1$:

$$C^s_{nm\zeta} = C^{s-\frac{1}{2}}_{n,m+\frac{1}{2}n,\zeta} + \sum_{\nu=-1}^{1} C^s_{n-1,m-s,\zeta+\nu} - \sum_{\nu=-1}^{1} C^s_{n-2,m-2s,\zeta+\nu} + C^s_{n-3,m-3s,\zeta}. \tag{23}$$

Undoubtedly this is not the only recurrence relation for $s'=\frac{1}{2}$, 1 and it is possible that there are more simple ones. Using (22) and (23) it is not difficult, however, to construct the tables of route numbers for the cases of N. 2 HO and N. 3 HO. In tables 1–4 we give the multiplicities q which were obtained with the help of (22), (23) and (17).

Table 1. Multiplicities $q_{nj\tau}$ for 3.2 HO. (In table 2 (from Bargmann and Moshinsky 1960) they are given up to $n = 12$).

n	τ	j = 0	1	2	3	4	5	6
1	$\frac{1}{2}$		1					
2	1	1		1				
	0		1					
3	$\frac{3}{2}$		1		1			
	$\frac{1}{2}$		1	1				
4	2	1		1		1		
	1		1	1	1			
	0	1		1				
5	$\frac{5}{2}$		1		1		1	
	$\frac{3}{2}$		1	1	1	1		
	$\frac{1}{2}$		1	1	1			
6	3	1		1		1		1
	2		1	1	1	1	1	
	1	1		2	1	1		
	0		1		1			

Table 2. Multiplicities $q_{nj\tau}$ for 4.2 HO.

n	τ	j = 0		1		2		3		4		5		6		7		8		9		10		11		12
1	$\frac{1}{2}$				1																					
2	1			1				1																		
	0	1				1																				
3	$\frac{3}{2}$				1		1				1															
	$\frac{1}{2}$		1		1		1		1																	
4	2	1				1		1		1				1												
	1			2		1		2		1		1														
	0	1				2				1																
5	$\frac{5}{2}$				1		1		1		1		1				1									
	$\frac{3}{2}$		1		2		2		2		2		1		1											
	$\frac{1}{2}$		1		2		2		2		1		1													
6	3			1				2		1		1		1		1				1						
	2	1		1		3		2		3		2		2		1		1								
	1			3		2		4		2		3		1		1										
	0	2				2		1		2				1												
7	$\frac{7}{2}$				1		1		1		2		1		1		1		1				1			
	$\frac{5}{2}$		1		2		3		3		3		3		2		2		1		1					
	$\frac{3}{2}$		2		3		4		4		4		3		3		1		1							
	$\frac{1}{2}$		1		3		3		3		4		2		1		1									
8	4	1				1		1		2		1		2		1		1		1		1				1
	3			2		2		4		3		4		3		3		2		2		1		1		
	2	2		2		5		4		6		4		5		3		3		1		1				
	1			4		3		6		4		5		3		3		1		1						
	0	2				3		1		3		1		2				1								

Table 3. Multiplicities $q_{nj\tau}$ for 3.3 HO.

n	τ	j = 0	1	2	3	4	5	6	7	8
1	1		1							
2	2	1		1						
	1		1							
	0	1		1						
3	3		1		1					
	2		1	1						
	1		2	1	1					
	0	1								
4	4	1		1		1				
	3		1	1	1					
	2	2	1	3	1	1				
	1		2	1	1					
	0	2		2		1				
5	5		1		1		1			
	4		1	1	1	1				
	3		3	2	3	1	1			
	2	1	2	3	2	1				
	1		4	2	3	1	1			
	0	1		1						
6	6	1		1		1		1		
	5		1	1	1	1	1			
	4	2	1	4	2	3	1	1		
	3	1	3	3	4	2	1			
	2	3	2	7	3	4	1	1		
	1		4	2	3	1	1			
	0	3		3	1	2		1		
7	7		1		1		1		1	
	6		1	1	1	1	1	1		
	5		3	2	4	2	3	1	1	
	4	1	3	4	4	4	2	1		
	3		6	5	8	4	4	1	1	
	2	2	4	6	5	4	2	1		
	1		6	4	6	3	3	1	1	
	0	2		2		1				
8	8	1		1		1		1		1
	7		1	1	1	1	1	1	1	
	6	2	1	4	2	4	2	3	1	1
	5	1	3	4	5	4	4	2	1	
	4	4	3	9	6	9	4	4	1	1
	3	1	6	7	9	6	5	2	1	
	2	5	4	11	7	9	4	4	1	1
	1		6	4	6	3	3	1	1	
	0	4		5	1	4	1	2		1

Table 4. Multiplicities $q_{nj\tau}$ for 4.3 HO.

n	τ	j																							
		0		1		2		3		4		5		6		7		8		9		10		11	12
1	1				1																				
2	2			1				1																	
	1	1				1																			
	0			1				1																	
3	3				1		1				1														
	2		1		1		1		1																
	1		1		2		2		1		1														
	0				1																				
4	4	1				1		1		1				1											
	3			2		1		2		1		1													
	2	2		2		4		3		3		1		1											
	1			3		2		3		1		1													
	0	2				3		1		2				1											
5	5				1		1		1		1		1				1								
	4		1		2		2		2		2		1		1										
	3		2		5		5		5		4		3		1		1								
	2		3		5		6		5		4		2		1										
	1		3		6		6		6		4		3		1		1								
	0		1		1		2		1		1														
6	6			1				2		1		1		1		1				1					
	5	1		1		3		2		3		2		2		1		1							
	4	1		5		5		8		6		6		4		3		1		1					
	3	4		5		9		9		9		6		5		2		1							
	2	1		10		10		14		10		10		5		4		1		1					
	1	4		4		10		7		9		4		4		1		1							
	0			5		2		7		3		4		2		2				1					
7	7				1		1		1		2		1		1		1		1				1		
	6		1		2		3		3		3		3		2		2		1		1				
	5		4		5		8		8		9		7		6		4		3		1		1		
	4		5		10		12		13		12		10		7		5		2		1				
	3		8		15		19		19		19		14		11		6		4		1		1		
	2		8		15		18		19		16		13		8		5		2		1				
	1		7		13		17		16		16		11		8		5		3		1		1		
	0		2		4		4		5		3		3		1		1								
8	8	1				1		1		2		1		2		1		1		1		1			1
	7			2		2		4		3		4		3		3		2		2		1		1	
	6	3		4		8		9		11		9		10		7		6		4		3		1	1
	5	2		10		12		17		16		16		13		11		7		5		2		1	
	4	8		13		24		25		29		24		23		15		12		6		4		1	1
	3	4		20		24		33		29		29		21		17		9		6		2		1	
	2	10		17		31		32		37		28		26		16		12		6		4		1	1
	1	2		16		17		25		20		21		13		11		5		4		1		1	
	0	6		3		12		8		14		7		10		4		5		2		2			1

In I the following property of the numbers C^s_{nm} ($s = 0$) was noted: for $n > n_0$ upper n_0+1 numbers $C^s_{n,ns}, C^s_{n,ns-1}, \ldots, C^s_{n,ns-n}$ are identical. There are two more general properties for the $N.N'$ HO which are true both for the route numbers C and for the multiplicities q.

Property 1. Let us take the rectangle of multiplicities for a given n_0 and for the definite $N.N'$. We draw a straight line cutting off n_0+1 numbers on the top table edge (from right to left) and as many numbers on the right table edge (from top to bottom) (see table 4). The numbers, which occur above this line, repeat in total in the upper right corners of all rectangles of multiplicities with $n > n_0$. This property appreciably simplifies the problem of multiplicity construction.

Property 2. It concerns a relation between tables with neighbouring numbers N or N' and with the same number n. We fix the latter number. Under such conditions the upper N' lines in the table $N.(N'+1)$ coincide with the upper lines in the table $N.N'$ and the right N columns in the table $(N+1).N'$ coincide with the right column in the table $N.N'$.

We may check the correctness of the q-tables with the help of dimension relations similar to (10). We seek the representation dimensions $M^{ss'}_n$ of an $N.N'$ HO system with fixed numbers n and τ. It is possible to find explicit formulae in some simple cases:

$$M^{ss'}_{n\tau} = \sum_m C^{ss'}_{nm\zeta}\Big|_{\zeta=\tau} = \sum_j (2j+1) q^{ss'}_{nj\tau} \tag{24}$$

$$M^{s,1/2}_{n\tau} = \binom{\frac{1}{2}n + N - 1 + \tau}{N-1}\binom{\frac{1}{2}n + N - 1 - \tau}{N-1} \tag{25}$$

$$M^{1/2,1}_{n\tau} = (i+2+2\tau)\binom{i+3}{3}, \qquad n-\tau = 2i+1, \tag{26}$$

$$M^{1/2,1}_{n\tau} = \tfrac{1}{3}[(i+1)(i+2)+1+(2i+3)\tau]\binom{i+2}{2}, \qquad n-\tau = 2i, \tag{27}$$

$$M^{1,1}_{n\tau} = \tfrac{1}{35}\left[4i^3+37i^2+113i+105+7(2i^2+15i+30)\tau+7(4i+15)\binom{\tau}{2}\right]\binom{i+4}{2},$$
$$n-\tau = 2i+1, \tag{28}$$

$$M^{1,1}_{n\tau} = \tfrac{1}{35}\left[4i^3+23i^2+43i+35+7(i+2)(2i+5)\tau+7(4i+5)\binom{\tau}{2}\right]\binom{i+4}{4},$$
$$n-\tau = 2i. \tag{29}$$

4. Multiplicities and dimensions in SU(N) $\supset$ O(3)

We now give a comparison of the multiplicities q (§§ 2,3) with the multiplicities which appear in the reduction SU(N)$\supset$O(3). Therefore we will briefly describe the reduction. Some of the structures and relations, outlined schematically in this section, are given at greater length in, e.g., Hamermesh (1962).

The IR $F^s_{[\lambda]}$ of SU(N) reduces to the sum of IR D_j of O(3) in the following way:

$$F^s_{[\lambda]} = \sum_j f^s_{[\lambda]j} D_j, \tag{30}$$

here $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_N]$, $\Sigma\, \lambda_i = n$, the multiplicities $f^s_{[\lambda]j}$ take integer values including zero. The dimensions $L^s_{[\lambda]}$ of IR $F^s_{[\lambda]}$ are defined by the well known Weyl formula

$$L^s_{[\lambda]} = \prod_{i>j}^{2s+1} \frac{\lambda_j - \lambda_i + i - j}{i - j}. \tag{31}$$

The reduction SU(N) ⊃ O(3) is closely related with the determination of the general AM of an n-atom system when each of these atoms has $2s+1 = N$ equidistant levels. Such systems may have AM $j = ns, ns-1, \ldots, 0$ (or $\frac{1}{2}$) each occurring P^s_{nj} times. The numbers P^s_{nj} were described earlier (Mikhailov 1977). They are the sums of all multiplicities $f^s_{[\lambda]j}$ for given j, s, n. The summation weights are the dimensions $N_{[\lambda]}$ of symmetric group IR with Young tableaux $[\lambda]$

$$P^s_{nj} = \sum_{[\lambda]} f^s_{[\lambda]j} N_{[\lambda]}. \tag{32}$$

Dimensions and multiplicities encountered in this reduction also satisfy the relations:

$$\sum_j (2j+1) f^s_{[\lambda]j} = L^s_{[\lambda]}, \tag{33}$$

$$\sum_{[\lambda]} L^s_{[\lambda]} N_{[\lambda]} = \sum_j (2j+1) P^s_{nj} = (2s+1)^n = N^n. \tag{34}$$

Table 5 illustrates (30)–(34) for two cases $n = 5$, $s = 1$ and $n = 5$, $s = \frac{3}{2}$.

A new simple graphical method for the determination of $f^1_{[\lambda]j}$, related to the Young tableaux, is given in the appendix.

Table 5. Numbers $f^s_{[\lambda]}$, $L^s_{[\lambda]}$, P^s_{nj} for $n = 5$, $s = 1, \frac{3}{2}$.

s	n	$[\lambda]$	$N_{[\lambda]}$	j									$L^s_{[\lambda]}$
				0	1	2	3	4	5	6	7	8	
1	5	[5]	1		1		1		1				21
		[4, 1]	4		1	1	1	1					24
		$[3, 2] = [3, 1]$	5		1	1	1						15
		$[3, 1^2] = [2]$	6	1		1							6
		$[2^2, 1] = [1]$	5		1								3
		P^s_{nj}		6	15	15	10	4	1				$243 = 3^5$
$\frac{3}{2}$	5	[5]	1		1	1	1	1	1		1		56
		[4, 1]	4	1	2	2	2	2	1	1			84
		[3, 2]	5	1	2	2	2	1	1				60
		$[3, 1^2]$	6	1	1	2	1	1					36
		$[2^2, 1] = [2, 1]$	5	1	1	1	1						20
		$[2, 1^3] = [1]$	4		1								4
		P^s_{nj}		20	34	36	30	20	10	4	1		$1024 = 4^5$

5. Relation between $N.N'$ HO and $SU(N) \supset O(3)$

From I we know that one N-dimensional HO supplies us with the AM multiplicities in symmetric IR (Young tableau $[\lambda_1, 0, 0, \ldots]$) of $SU(N)$. From tables 1–4 and Bargmann and Moshinsky (1960) it is easy to determine that two N-dimensional HO give the AM multiplicities in IR $[\lambda_1, 0, 0, \ldots, 0]$ and $[\lambda_1, 1, 0, 0, \ldots, 0]$. There is the simple correspondence

$$q^{s,1/2}_{nj\tau} = f^s_{[\lambda]j} \qquad \begin{cases} \text{(i)} & [\lambda] = [n, 0, 0, \ldots], \qquad \tau = \tfrac{1}{2}n \\ \text{(ii)} & [\lambda] = [n-1, 1, 0, 0, \ldots], \qquad \tau = \tfrac{1}{2}n - 1. \end{cases} \tag{35}$$

One might expect that the N' N-dimensional HO will give information for the determination of AM multiplicities in IR with more general Young tableaux $[\lambda_1, \lambda_2, \ldots, \lambda_{N'}, 0, 0, \ldots]$. Indeed, detailed analysis shows that the numbers q are the quadratic contraction of the numbers f:

$$q^{ss'}_{nj\tau} = \sum_{[\lambda]} f^s_{[\lambda]j} f^{s'}_{[\lambda]\tau}. \tag{36}$$

Here the summation extends over all possible partitions $[\lambda]$, existing for given s, s' and n.

Having written the two matrices of numbers f in table 5 we can multiply them in accordance with (36). Preliminarily, it is necessary to discard from numbers $f^{3/2}$ the line $[\lambda] = [2, 1^3]$ because a four-line tableau is not possible for $s = 1$ ($N = 3$). The result of the multiplication is a matrix of numbers q from table 4 (when $n = 5$). Similarly, every matrix of numbers q from tables 1–4 can be expressed in the form (36).

The dimensions $L^s_{[\lambda]}$ obey a relation which is a consequence of (36). To obtain this we write down the relation, supplementing (30), for another group $SU(2s'+1)$ and the same partition $[\lambda]$:

$$F^{s'}_{[\lambda]} = \sum_{\tau} f^{s'}_{[\lambda]\tau} D_\tau. \tag{37}$$

Now we multiply the right and left sides of (30) and (37) separately and carry out the sum over all partitions $[\lambda]$:

$$\sum_{[\lambda]} F^s_{[\lambda]} F^{s'}_{[\lambda]} = \sum_{j,\tau} \left(\sum_{[\lambda]} f^s_{[\lambda]j} f^{s'}_{[\lambda]\tau} \right) D_j D_\tau = \sum_{j,\tau} q^{ss'}_{nj\tau} D_{j\tau}. \tag{38}$$

It is known that the dimension of the last term, in accordance with (8) and (9), is equal to $M^{ss'}_n$. Therefore

$$\sum_{[\lambda]} L^s_{[\lambda]} L^{s'}_{[\lambda]} = \binom{n + N.N' - 1}{n}. \tag{39}$$

6. Conclusions

The method of construction of AM multiplicities in $N.N'$ HO (§§ 2,3) and the relation of numbers q with numbers f (36) may, in principle, be used as an alternative to the Jahn method. If we know $q^{ss'}$ and $f^{s'}$ we will easily get f^s with the help of (36).

Moreover, if we do not know either f^s or $f^{s'}$ we may, beginning from (35), get subsequently all the lines $[\lambda]$ of numbers f^s.

Construction of numbers f by this method has several stages of calculation: finding the numbers C, then the numbers q and finally the numbers f. This is its shortcoming. However, the advantage of this method is that we have here a comparatively simple generating function and recurrence relations which completely solve the problem. The trees graphically illustrate the essence of the method.

The result of this paper allows generalisation to the case when the dimension of the HO is the product of an arbitrary number of integers, i.e. in terms of our notation to the case $N_1 N_2 \ldots N_k$ HO.

The approach to the related problems of HO and $SU(N) \supset O(3)$ is restricted here by the question of dimensions and multiplicities. But it may probably be extended to the other group properties such as the construction of operators, basis states, matrix elements, etc.

Appendix

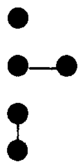
The AM content of SU(3) IR is already well known (Elliot 1958a,b, Bargmann and Moshinsky 1960, 1961). However, until recently there was no simple graphical method for calculating AM multiplicities. Recently, Hughes (1973b) suggested a method connected with diagrams. Here we present a graphical method related to the Young tableaux.

We will draw the Young tableaux as rows of dots. Because of $[\lambda_1, \lambda_2, \lambda_3] = [\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0]$ we consider only two-row tableaux. We introduce three elemental dot combinations to which we assign the definite AM:

solitary spot $j=1$

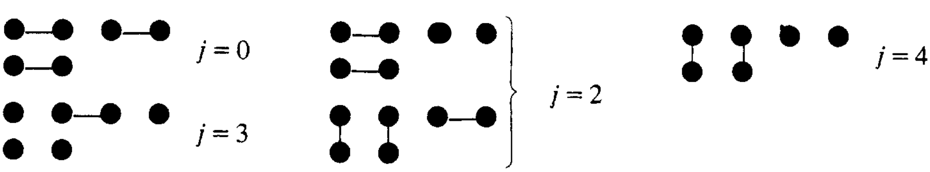
horizontal couple $j=0$

vertical couple $j=1$.



We must fill the definite Young tableau in accordance with the following rule. The solitary spot may be: (i) only in the first row excess; and (ii) in the rightmost dot of the second row. The horizontal and vertical couples may be in all possible positions. After the occupation of the tableau we calculate the total AM of the tableau by simple summation of all elemental AM. The number of different ways of the occupation with definite AM is the multiplicity to be determined.

Let us consider an example $[\lambda] = [5, 3, 1] = [4, 2]$



Therefore IR $F^1_{[\lambda]}$, $[\lambda] = [5, 3, 1]$, decompose into IR D_j by the following way:

$$F^1_{[5,3,1]} = D_0 + 2D_2 + D_3 + D_4.$$

It is not only the visual demonstration that is the advantage of this graphical method. Practically, it removes degeneracy of the AM states in SU(3) IR: the states may be labelled with the Young tableaux which are filled by the elemental combinations in a definite way.

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